

EMBEDDED CONSTANT CURVATURE CURVES ON CONVEX SURFACES

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ABSTRACT. We prove the existence of embedded closed constant curvature curves on convex surfaces.

1. INTRODUCTION

Let (S^2, g) be a two dimensional oriented sphere with a smooth Riemannian metric g . We prove existence results for closed embedded curves with prescribed geodesic curvature in (S^2, g) , when the Gauss curvature K_g of the metric g is positive. In particular, we study the existence of closed embedded constant curvature curves on strictly convex spheres.

Let $c : S^2 \rightarrow \mathbb{R}$ be a smooth positive function. We consider the following equation for curves γ on S^2 :

$$D_{t,g}\dot{\gamma}(t) = |\dot{\gamma}(t)|_g c(\gamma(t)) J_g(\gamma(t)) \dot{\gamma}(t), \quad (1.1)$$

where $D_{t,g}$ is the covariant derivative with respect to g , and $J_g(x)$ is the rotation by $\pi/2$ in $T_x S^2$ with respect to g and the given orientation. Solutions γ to equation (1.1) are constant speed curves with geodesic curvature $c_g(\gamma, t)$ given by $c(\gamma(t))$. We remark that, besides the geometric interpretation, (1.1) describes the motion of a charged particle on (S^2, g) in a magnetic field with magnetic form $c\mu_g$, where μ_g denotes the volume form of g (see [1–3]).

From [8, 9] closed embedded solutions to (1.1) exist, if the curvature function c is large enough depending on the metric g . If g is $\frac{1}{4}$ -pinched, i.e. $\sup K_g < 4 \inf K_g$, then there are embedded closed solutions of (1.1) for every positive function c (see [6, 8]). It is conjectured in [3, §5]) and [6] that this remains true for an arbitrary metric g on S^2 . We note that, if K_g and c are positive, then from [5–7] there are always Alexandrov embedded, closed solutions to (1.1), i.e. curves that bound an immersed disc.

We shall show that on strictly convex spheres, i.e. $K_g > 0$, there are closed embedded solutions to (1.1), if the curvature function is small enough depending on the metric g . In particular, we show

Theorem 1.1. *Suppose (S^2, g) has positive Gauss curvature. Then there is $\varepsilon_0 > 0$ such that for all $0 < c \leq \varepsilon_0$ there are two embedded closed curves with constant geodesic curvature c .*

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Hence, on strictly convex spheres there are closed, embedded constant curvature curves for large and small values of $c > 0$. We expect that this is true for all $c > 0$.

We use the degree theory developed in [8] to prove our existence result. The required compactness results are given in section 2. The a priori estimates follow from Reilly's formula. The fact that a geodesic cannot touch itself continues to hold for solutions to (1.1) when the geodesic curvature is close to zero. This allows to carry out the degree argument within the class of embedded curves. The existence result is given in section 3.

2. THE APRIORI ESTIMATE

Lemma 2.1. *Suppose (S^2, g) has positive Gauss curvature K_g and $\gamma \in C^2(S^1, S^2)$ is an (Alexandrov) embedded curve with nonnegative geodesic curvature. Then the length $L(\gamma)$ of γ is bounded by*

$$L(\gamma) \leq 2\pi\sqrt{2}\left(\inf_{S^2} K_g\right)^{-\frac{1}{2}}.$$

Proof. We use Reilly's formula [4]: Let (M, g) be a compact Riemannian manifold with boundary ∂M , $f \in C^\infty(M)$, $z = f|_{\partial M}$ and $u = \frac{\partial f}{\partial n}$ on ∂M , where n denotes the outer normal. Then

$$\begin{aligned} \int_M (\bar{\Delta}f)^2 - |\bar{\nabla}^2 f|^2 &= \int_M \text{Ric}(\bar{\nabla}f, \bar{\nabla}f) \\ &\quad + \int_{\partial M} (\Delta z + Hu)u - \langle \nabla z, \nabla u \rangle + \Pi(\nabla z, \nabla z), \end{aligned} \quad (2.1)$$

where we denote by $\bar{\Delta}$, Δ and $\bar{\nabla}$, ∇ the Laplacians and covariant derivatives on M and ∂M respectively; H is the mean curvature and Π is the second fundamental form of ∂M .

If the curve γ is embedded or Alexandrov embedded, then we may assume that we are in the above situation with $\partial M = \gamma$.

We take z an eigenfunction of λ_1 the first nontrivial eigenvalue on ∂M

$$\Delta z + \lambda_1 z = 0 \text{ on } \partial M,$$

and f its harmonic extension to M . In dimension two, (2.1) leads to

$$\int_M (\bar{\Delta}f)^2 - |\bar{\nabla}^2 f|^2 = \int_M K_g |\bar{\nabla}f|^2 + \int_{\partial M} \Delta z u + cu^2 - \langle \nabla z, \nabla u \rangle + c|\nabla z|^2,$$

where c is the geodesic curvature of ∂M and K_g denotes the Gauss curvature of M . Using the fact that the geodesic curvature c of ∂M is nonnegative, f is harmonic, and z is an eigenfunction, we obtain

$$0 \geq \left(\inf_M K_g\right) \int_M |\bar{\nabla}f|^2 - 2\lambda_1 \int_{\partial M} zu.$$

Integrating by parts again we see

$$\int_{\partial M} zu = \int_M |\bar{\nabla}f|^2 + f\bar{\Delta}f = \int_M |\bar{\nabla}f|^2.$$

Since z is a nontrivial eigenfunction, f is non constant such that we arrive at

$$\left(\inf_M K_g\right) \leq 2\lambda_1.$$

The first nontrivial eigenvalue λ_1 depends only on the length $L(\partial\Omega)$ of ∂M and is given by

$$\lambda_1 = \frac{4\pi^2}{L(\partial\Omega)^2}.$$

This gives the claim. \square

Lemma 2.2. *Let (γ_n) be a sequence of simple closed curves converging in $C^2(S^1, S^2)$ to a non constant closed geodesic γ in (S^2, g) . Then γ is simple as well.*

Proof. To obtain a contradiction assume that there are $\theta_1 \neq \theta_2$ in $S^1 = \mathbb{R}/\mathbb{Z}$ such that $\gamma(\theta_1) = \gamma(\theta_2)$. Since γ is a limit of simple curves and $|\dot{\gamma}| \equiv \text{const}$, there holds

$$\dot{\gamma}(\theta_1) = \pm \dot{\gamma}(\theta_2).$$

From the uniqueness of geodesics we have for $t \in S^1$

$$\gamma(t) = \gamma(\pm(t - \theta_1) + \theta_2).$$

Setting $t = (\theta_1 + \theta_2)/2$ we find that

$$\gamma(t) = \gamma(t - \theta_1 + \theta_2).$$

Consequently, γ is a n -fold covering of a simple geodesic for some $n \geq 2$. From the stability of the winding number, we get a contradiction. \square

We denote by g_{can} the standard round metric on S^2 with curvature $K_{g_{can}} \equiv 1$. We fix a function $\varphi \in C^\infty(S^2, \mathbb{R})$ and a conformal metric

$$g = e^\varphi g_{can}.$$

on S^2 with positive Gauss curvature $K_g > 0$. We consider the family of metrics $\{g_t : t \in [0, 1]\}$ defined by

$$g_t := e^{t\varphi} g_{can}.$$

Then the Gauss curvature K_{g_t} of the metric g_t satisfies for some $K_0 > 0$

$$\begin{aligned} K_{g_t} &= e^{-t\varphi} (-t\Delta_{g_{can}}(\varphi) + 2) \\ &= e^{-t\varphi} (-t(2 - K_g e^\varphi) + 2) \geq K_0, \end{aligned}$$

because K_g is positive.

Lemma 2.3. *Suppose $c : S^2 \rightarrow \mathbb{R}$ is a nonnegative smooth function. For $r \in [0, 1]$ we define the set of closed curves \mathcal{M}_r by*

$$\begin{aligned} \mathcal{M}_r &:= \{\gamma \in C^2(S^1, S^2) : \gamma \text{ is embedded, } |\dot{\gamma}|_g \equiv \text{const}, \\ &\quad \exists(t, s) \in [0, 1] \times [0, r] : c_{g_t}(\gamma, \theta) = sc(\gamma(\theta)) \forall \theta \in S^1.\}, \end{aligned}$$

where $c_{g_t}(\gamma, \cdot)$ denotes the geodesic curvature of γ with respect to g_t . Then there is $\varepsilon_0 > 0$, such that $\mathcal{M}_{\varepsilon_0}$ is compact. Moreover, $\varepsilon_0 > 0$ may be chosen uniformly with respect to $\|c\|_\infty$.

Proof. Let $(\gamma_n)_{n \in \mathbb{N}}$ be a sequence in \mathcal{M}_r for some $r > 0$. By Lemma 2.1 and (1.1) we get a uniform bound in $C^3(S^1, S^2)$ and from the Gauss-Bonnet formula the length of γ_n is bounded below, in both cases the bounds are uniform with respect to r . Since the metrics $\{g_t : t \in [0, 1]\}$ are uniformly equivalent, there is $C_0 > 0$ such that we have for all $t \in [0, 1]$

$$|\dot{\gamma}_n|_{g_t} > (C_0)^{-1} \text{ and } \|\gamma_n\|_{C^3(S^1, S^2), g_t} < C_0. \quad (2.2)$$

Up to a subsequence we may assume $(t_n, s_n) \rightarrow (t, s) \in [0, 1] \times [0, r]$,

$$\gamma_n \rightarrow \gamma \text{ in } C^2(S^1, S^2),$$

where $|\dot{\gamma}|_{g_t} \equiv \text{const}$ and

$$c_{g_t}(\gamma, \theta) = sc(\gamma(\theta)) \quad \forall \theta \in S^1. \quad (2.3)$$

Consequently, if \mathcal{M}_r is not compact, there is $(t, s) \in [0, 1] \times [0, r]$ and $\gamma_r \in C^2(S^1, S^2)$ satisfying $|\dot{\gamma}|_{g_t} \equiv \text{const}$ and (2.3), which is not embedded, but a limit of embedded curves in \mathcal{M}_r . Thus there are $\theta_1, \theta_2 \in S^1$, such that $\theta_1 \neq \theta_2$ and $\gamma_r(\theta_1) = \gamma_r(\theta_2)$. From (2.2) we deduce that there is $\delta > 0$ independent of r , such that

$$\delta \leq |\theta_1 - \theta_2| \leq 1 - \delta. \quad (2.4)$$

Hence for any $n \in \mathbb{N}$ there is $\gamma_n \in \mathcal{M}_r$ such that

$$\text{dist}(\gamma_n(\theta_1), \gamma_n(\theta_2)) \leq \frac{1}{n}. \quad (2.5)$$

To obtain a contradiction assume there is (r_n) converging to 0 such that \mathcal{M}_{r_n} is not compact. Then for any $n \in \mathbb{N}$ there are $(t_n, s_n) \in [0, 1] \times [0, r_n]$, $\theta_{1,n}, \theta_{2,n} \in S^1$, and $\gamma_n \in \mathcal{M}_{r_n}$ that satisfy (2.4) and (2.5). From the uniform bounds, going to a subsequence, we may assume that $(t_n, s_n, \gamma_n, \theta_{1,n}, \theta_{2,n})$ converge to $(t, 0, \gamma, \theta_1, \theta_2)$, where θ_1 and θ_2 satisfy (2.4) and γ is a closed nontrivial geodesic in (S^2, g_t) satisfying $\gamma(\theta_1) = \gamma(\theta_2)$. This contradicts Lemma 2.2. Since all the above bounds are uniform with respect to $\|c\|_\infty$, the constant $\varepsilon_0 > 0$ may be chosen uniform with respect to $\|c\|_\infty$ as well. \square

3. EXISTENCE RESULTS

We follow [8] and consider solutions to (2.3) as zeros of the vector field $X_{c,g}$ defined on the Sobolev space $H^{2,2}(S^1, S^2)$ as follows: For $\gamma \in H^{2,2}(S^1, S^2)$ we let $X_{c,g}(\gamma)$ be the unique weak solution of

$$(-D_{t,g}^2 + 1)X_{c,g}(\gamma) = -D_{t,g}\dot{\gamma} + |\dot{\gamma}|_g c(\gamma)J_g(\gamma)\dot{\gamma} \quad (3.1)$$

in $T_\gamma H^{2,2}(S^1, S^2)$.

Solutions to (2.3) or equivalently zeros of $X_{c,g}$ are invariant under a circle

action: For $\theta \in S^1 = \mathbb{R}/\mathbb{Z}$ and $\gamma \in H^{2,2}(S^1, S^2)$ we define $\theta * \gamma \in H^{2,2}(S^1, S^2)$ by

$$\theta * \gamma(t) = \gamma(t + \theta).$$

Thus, any solution gives rise to a S^1 -orbit of solutions and we say that two solutions γ_1 and γ_2 are (geometrically) distinct, if $S^1 * \gamma_1 \neq S^1 * \gamma_2$. We denote by $M \subset H^{2,2}(S^1, S^2)$ the set

$$M := \{\gamma \in H^{2,2}(S^1, S^2) : \dot{\gamma}(\theta) \neq 0 \forall \theta \in S^1 \text{ and } \gamma \text{ is embedded.}\}.$$

In [8] an integer valued S^1 -degree, $\chi_{S^1}(X_{c,g}, M)$ is introduced. The S^1 -degree is defined, whenever $X_{c,g}$ is proper in M , i.e. the set $\{\gamma \in M : X_{c,g}(\gamma) = 0\}$ is compact, and does not change under homotopies in the class of proper vector fields.

Theorem 3.1. *Suppose (S^2, g) has positive Gauss curvature. Then there is $\varepsilon_0 > 0$ such that for all smooth functions $c : S^2 \rightarrow \mathbb{R}$ satisfying $0 < c \leq \varepsilon_0$ there are two embedded geometrically distinct closed curves which solve equation (1.1).*

Proof. From the uniformization theorem up to isometries we may assume without loss of generality that

$$g = e^\varphi g_{can},$$

where $\varphi \in C^\infty(S^2, \mathbb{R})$ and g_{can} denotes the standard round metric on S^2 . We consider the set of metrics $\{g_t : t \in [0, 1]\}$ defined by

$$g_t := e^{t\varphi} g_{can}.$$

From Lemma 2.3 there is $\varepsilon_0 > 0$ such that the set

$$\{\gamma \in M : X_{c,g_t}(\gamma) = 0 \text{ for some } t \in [0, 1]\}$$

is compact for all functions c with $0 < c \leq \varepsilon_0$. Consequently,

$$[0, 1] \ni t \mapsto X_{c,g_t}$$

is a homotopy of proper vector fields. From [8] there holds

$$-2 = \chi_{S^1}(X_{c,g_{can}}, M),$$

such that the homotopy invariance leads to

$$\chi_{S^1}(X_{c,g}, M) = -2$$

Since the local degree of an isolated zero orbit is greater than or equal to -1 by [8, Lem 4.1], there are at least two geometrically distinct solutions to (1.1). This gives the claim. \square

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